Pergamon

Computers Math. Applic. Vol. 34, No. 2–4, pp. 103–109, 1997

0898-1221/97 $17.00 + 0.00

PII: S0898-1221(97)00120-X

Jordan Blocks in a One-Dimensional Markov Map

D. J. DRIEBE
Center for Studies in Statistical Mechanics and Complex Systems
The University of Texas at Austin, Austin, TX 78712, U.S.A.
and
International Solvay Institutes for Physics and Chemistry
Free University of Brussels, 1050 Brussels, Belgium

Abstract—A simple one-dimensional chaotic map, whose spectral decomposition of the Frobenius-Perron operator in the space of polynomials has Jordan block structure is given and analyzed. The Jordan block structure implies that some correlation functions have exponential decay modified by coefficients, which are polynomials in t. Such a correlation is given for this system and a numerical simulation, which agrees with the theoretical prediction is presented.

Keywords—Frobenius-Perron operator, Jordan blocks, Modified exponential decay, Markov map.

1. INTRODUCTION

Chaotic maps have proven to be very useful as models of unstable dynamics. Since for a discrete-time process only one degree of freedom is necessary for chaos, the models may be very simple and sometimes amenable to complete analysis. Because of trajectory instability, it is quite natural to consider the evolution of probability densities in chaotic systems. The time evolution operator for probability densities in chaotic maps is known as the Frobenius-Perron operator [1]. The approach to equilibrium of observables and correlation functions may then be studied from the spectral properties of the Frobenius-Perron operator.

For a class of observables, many fully chaotic one-dimensional maps display pure exponential decay. This corresponds to simple eigenvalues of the Frobenius-Perron operator and thus, simple poles of its resolvent. Associated with the simple poles, one can construct eigenstates. For some maps, the complete spectral decomposition of the Frobenius-Perron operator has been obtained and has elements (the left-eigenvectors) in a generalized functional space [2–5].

The simplest fully chaotic one-dimensional maps are piecewise-linear maps. If the linear branches all map onto the unit interval, on which the full map is defined, then the map is Markov. In this case, the x auto-correlation function always displays either pure exponential decay or is delta correlated [6]. We may ask about the behavior of more general correlation functions, where the observables are arbitrary polynomials. If the Frobenius-Perron operator

I thank I. Prigogine for his support and encouragement. I am grateful to D. Daems for his assistance in performing the numerical simulation. I thank O. Bandtlow for bringing reference [8] to my attention and P. Gaspard and F. Bosco for their suggestions. I acknowledge U.S. Department of Energy Grant No. FG03-94ER14465, the Robert A. Welch Foundation Grant No. F-0365, and the European Communities Commission (Contract No. 27155.1/BAS) for support of this work.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX is completely diagonalizable in the space of polynomials, then correlations among polynomials will always be a sum of exponentially decaying contributions. If the operator is only reducible to a Jordan block form, then the exponential decay will be modified by coefficients which are polynomials in the time variable.

Some two-dimensional maps, including the Baker map, have been shown to have a Jordan block generalized spectral decomposition [7]. Until now, no one-dimensional maps with such structure have been studied. In fact, piecewise-linear Markov maps are a special case of a class of maps for which Mayer and Roepstorff [8] have conjectured, have only simple eigenvalues and are thus completely diagonalizable. The map studied in this paper is thus a counterexample to this conjecture.

2.A SIMPLE MAP WITH JORDAN BLOCKS

Consider the three-branch piecewise-linear map defined on the unit interval as

$$S(x) = \begin{cases} 3x, & 0 \le x < \dfrac{1}{3}, \\ 3x-1, & \dfrac{1}{3} \le x < \dfrac{2}{3}, \\ 3-3x, & \dfrac{2}{3} \le x \le 1. \end{cases} \tag{1}$$

The map is shown in Figure 1. Each branch maps onto the whole unit interval, so the map is Markov, and the absolute value of the slope of all three of the branches has the same value of 3. This uniform stretching factor means that the Lyapunov exponent of the map is $\log 3$.

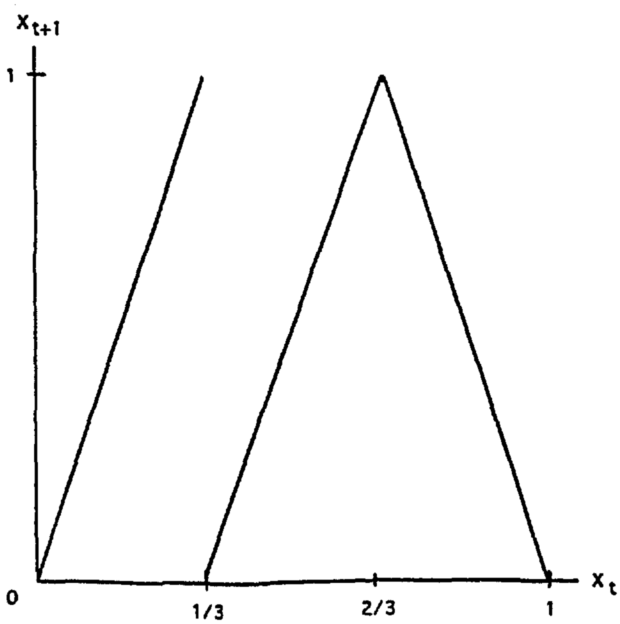

Figure 1. The three-branch map with Jordan blocks.

The Frobenius-Perron operator for this map is given by

$$U\rho(x) = \frac{1}{3}\left[\rho\left(\frac{x}{3}\right) + \rho\left(\frac{1+x}{3}\right) + \rho\left(\frac{3-x}{3}\right)\right],$$

where $\rho(x)$ is a density with respect to Lebesgue measure. It is clear that U admits polynomial eigenstates and by considering its action on monomials, it becomes a triangular matrix, so the eigenvalues are found along the diagonal. Except for the first eigenvalue of 1, the diagonal values are found to be two-fold degenerate. The rows and columns of the matrix are labeled beginning with zero so as to correspond with the associated polynomial. The diagonal elements at rows $2n-1$ and $2n$, where $n \geq 1$, both have the values of 3^{-2n}.

A suitable biorthonormal basis to begin the construction of the spectral representation of U is the basis formed from the eigenstates of the dyadic Bernoulli map. The right-states are the Bernoulli polynomials $B_n(x)$, defined by the generating function

$$\frac{pe^{xp}}{e^p-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} p^n.$$

The left-states are the generalized functions

$$\tilde{B}_n(x) = \frac{(-1)^{n-1}}{n!} \left[\delta^{(n-1)}(x-1) - \delta^{(n-1)}(x)\right].$$

These are the duals of the Bernoulli polynomials as

$$\langle \tilde{B}_n \mid B_m \rangle \equiv \int_0^1 dx\, \tilde{B}_n^*(x) B_m(x) = \delta_{nm},$$

where integration with respect to Lebesgue measure over the unit interval defines the inner product.

The matrix elements of U in the Bernoulli basis are

$$\langle \tilde{B}_n | U | B_m \rangle = \int_0^1 dx\, \tilde{B}_n U B_m(x) = \frac{m!}{n!(m-n)!} \frac{1}{3^n} I_{n,m},$$

where

$$I_{n,m} \equiv \int_0^{2/3} dx B_{m-n}(x) + (-1)^n \int_{2/3}^1 dx B_{m-n}(x).$$

If $n = m$, then $I_{n,n} = 1$ for n even and $I_{n,n} = 1/3$ for n odd. Thus, the diagonal elements of U in this basis are

$$e^{-\gamma_n} \equiv \langle \tilde{B}_n | U | B_n \rangle = \begin{cases} \dfrac{1}{3^n}, & n \text{ even}, \\ \dfrac{1}{3^{n+1}}, & n \text{ odd}, \end{cases}$$

where we denote the eigenvalue so as to stress the exponential decay with decay rate $\gamma_n = n \log 3$ if n is even and $\gamma_n = (n+1) \log 3$ if n is odd. The off-diagonal elements with $n < m$ are

$$\langle \tilde{B}_n | U | B_m \rangle = \begin{cases} 0, & n \text{ even}, \\ \dfrac{1}{m-n+1} \left[2B_{m-n+1}\left(\dfrac{2}{3}\right) - B_{m-n+1}(0) - B_{m-n+1}(1)\right], & n \text{ odd}. \end{cases}$$

Hence, U is upper-triangular in this basis with even-numbered rows having only the diagonal element nonvanishing.

We now decompose U into its diagonal part U_0 and off-diagonal part δU as

$$U = U_0 + \delta U,$$

where

$$\begin{aligned} U_0 &\equiv \sum_n |B_n\rangle \langle \tilde{B}_n | U | B_n \rangle \langle \tilde{B}_n| \\ &= |B_0\rangle \langle \tilde{B}_0| + \sum_{n=1}^{\infty} \frac{1}{3^{2n}} [|B_{2n-1}\rangle \langle \tilde{B}_{2n-1}| + |B_{2n}\rangle \langle \tilde{B}_{2n}|] \end{aligned}$$

and

$$\delta U \equiv \sum_{n<m} |B_n\rangle \langle \tilde{B}_n|U|B_m\rangle\langle \tilde{B}_m|,$$

where $n < m$ due to the upper-triangularity of U.

Because of the two-fold degeneracy of the diagonal elements, the resolvent of U may have double poles. We consider the analytic structure of the resolvent by looking at its matrix elements in the Bernoulli basis. It is convenient to expand the resolvent as

$$\langle \tilde{B}_m|\frac{1}{z-U}|B_{m'}\rangle = \sum_{n=0}^{\infty}\langle \tilde{B}_m|\frac{1}{z-U_0}\left(\delta U\frac{1}{z-U_0}\right)^n|B_{m'}\rangle. \tag{2}$$

Here we do not need to worry about the convergence of the expansion since all matrix elements $\langle \tilde{B}_m|\delta U^n|B_{m'}\rangle$ vanish for $n > m' - m$ due to the zero diagonal and upper triangularity of δU. In this way, the series in (2) terminates and nonvanishing terms have the form

$$\begin{aligned}
&\langle \tilde{B}_m|\frac{1}{z-U_0}\left(\delta U\frac{1}{z-U_0}\right)^n|B_{m'}\rangle \\
&= \sum_{l_1,l_2\ldots l_n}\langle \tilde{B}_m|\frac{1}{z-U_0}\delta U|B_{l_1}\rangle\langle \tilde{B}_{l_1}|\frac{1}{z-U_0}\delta U|B_{l_2}\rangle\langle \tilde{B}_{l_2}|\cdots\delta U|B_{l_{n-1}}\rangle \\
&\quad\times\langle \tilde{B}_{l_{n-1}}|\frac{1}{z-U_0}\delta U|B_{l_n}\rangle\langle \tilde{B}_{l_n}|\frac{1}{z-U_0}|B_{m'}\rangle \\
&= \sum_{l_1<l_2<\cdots<l_{n-1}}\frac{1}{z-e^{-\gamma_m}}\langle \tilde{B}_m|\delta U|B_{l_1}\rangle\frac{1}{z-e^{-\gamma_{l_1}}}\langle \tilde{B}_{l_1}|\delta U|B_{l_2}\rangle\cdots \\
&\quad\times\delta U|B_{l_{n-1}}\rangle\frac{1}{z-e^{-\gamma_{l_{n-1}}}}\langle \tilde{B}_{l_{n-1}}|\delta U|B_{m'}\rangle\frac{1}{z-e^{-\gamma_{m'}}}.
\end{aligned} \tag{3}$$

We have in (3) that $l_1 < l_2 < \cdots < l_{n-1}$, due to the strict upper-triangularity of δU. Since there is a two-fold multiplicity of the eigenvalues of U_0, the only way we may have a double pole in (3) is from transitions in the expansion from one even-order state to a dual state one unit lower such as $\langle \tilde{B}_{2k-1}|\delta U|B_{2k}\rangle$. But, then in general, to the right, the term $\langle \tilde{B}_{2k}|\delta U|B_l\rangle$ occurs. This term vanishes though, since $\langle \tilde{B}_n|U|B_m\rangle = 0$ if n is even. We do identify a double pole in (3) though if m' is even and $l_{n-1} = m' - 1$.

The double poles in the resolvent imply that U cannot be completely diagonalized in this basis but is only reducible to Jordan block form, with 2×2 blocks except for the first eigenvalue of one. The spectral decomposition of U is formally thus,

$$U = |\varphi_0\rangle\langle\tilde{\varphi}_0| + \sum_{k=1}^{\infty}\frac{1}{3^{2k}}\left[|\varphi_{2k-1}\rangle\langle\tilde{\varphi}_{2k-1}| + |\varphi_{2k}\rangle\langle\tilde{\varphi}_{2k}|\right] + |\varphi_{2k-1}\rangle\langle\tilde{\varphi}_{2k}|.$$

The right-principal vectors satisfy

$$\begin{aligned}
U|\varphi_{2k-1}\rangle &= \frac{1}{3^{2k}}|\varphi_{2k-1}\rangle, \\
U|\varphi_{2k}\rangle &= \frac{1}{3^{2k}}|\varphi_{2k}\rangle + |\varphi_{2k-1}\rangle.
\end{aligned}$$

The left-principal vectors satisfy

$$\begin{aligned}
\langle\tilde{\varphi}_{2k-1}|U &= \frac{1}{3^{2k}}\langle\tilde{\varphi}_{2k-1}| + \langle\tilde{\varphi}_{2k}|, \\
\langle\tilde{\varphi}_{2k}|U &= \frac{1}{3^{2k}}\langle\tilde{\varphi}_{2k}|.
\end{aligned}$$

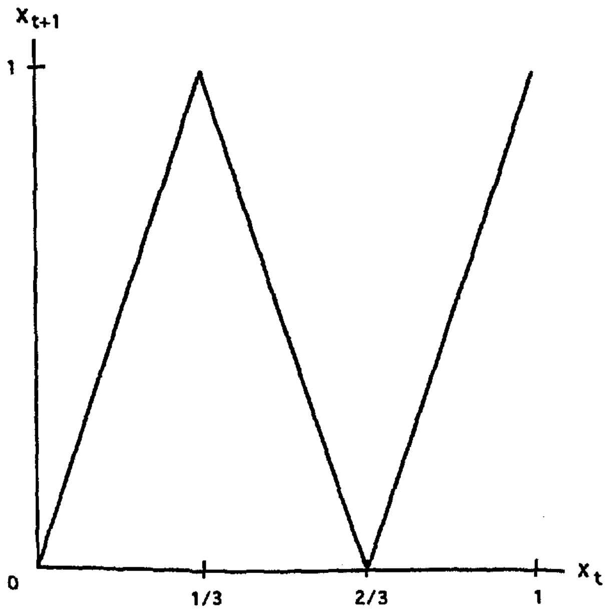

Figure 2. The three-branch, diagonalizable, incomplete tent map.

The principal vectors form a complete biorthonormal set as

$$\langle \tilde{\varphi}_n \mid \varphi_m \rangle = \delta_{n,m}$$

and

$$\sum_{n=0}^{\infty} |\varphi_n\rangle \langle \tilde{\varphi}_n| = I.$$

A general expression for the principal vectors may be obtained from a systematic method to evaluate the pole contributions from the resolvent. This calculation will be presented elsewhere [9].

We may easily determine the first few right-states through a direct method. The first four, with the coefficient of the highest power of x in the odd-order eigenpolynomials taken to be 1 are

$$\begin{aligned}
\varphi_0(x) &= 1, \\
\varphi_1(x) &= x - \frac{1}{2}, \\
\varphi_2(x) &= -\frac{27}{4}\left(x^2 - \frac{2}{3}x\right), \\
\varphi_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{3}{4}x - \frac{1}{8}.
\end{aligned}$$

To the even-order right principal vectors may be added, any multiple of the odd-order eigenstate with the same eigenvalue.

It is interesting to compare the map presented above with the three-branch incomplete tent map defined by

$$S_{T3}(x) = \begin{cases} 3x, & 0 \le x < \dfrac{1}{3}, \\ 2 - 3x, & \dfrac{1}{3} \le x < \dfrac{2}{3}, \\ 3x - 2, & \dfrac{2}{3} \le x \le 1. \end{cases}$$

The map is shown in Figure 2. The associated Frobenius-Perron operator is

$$U_{T3}\rho(x) = \frac{1}{3}\left[\rho\left(\frac{x}{3}\right) + \rho\left(\frac{2-x}{3}\right) + \rho\left(\frac{2+x}{3}\right)\right].$$

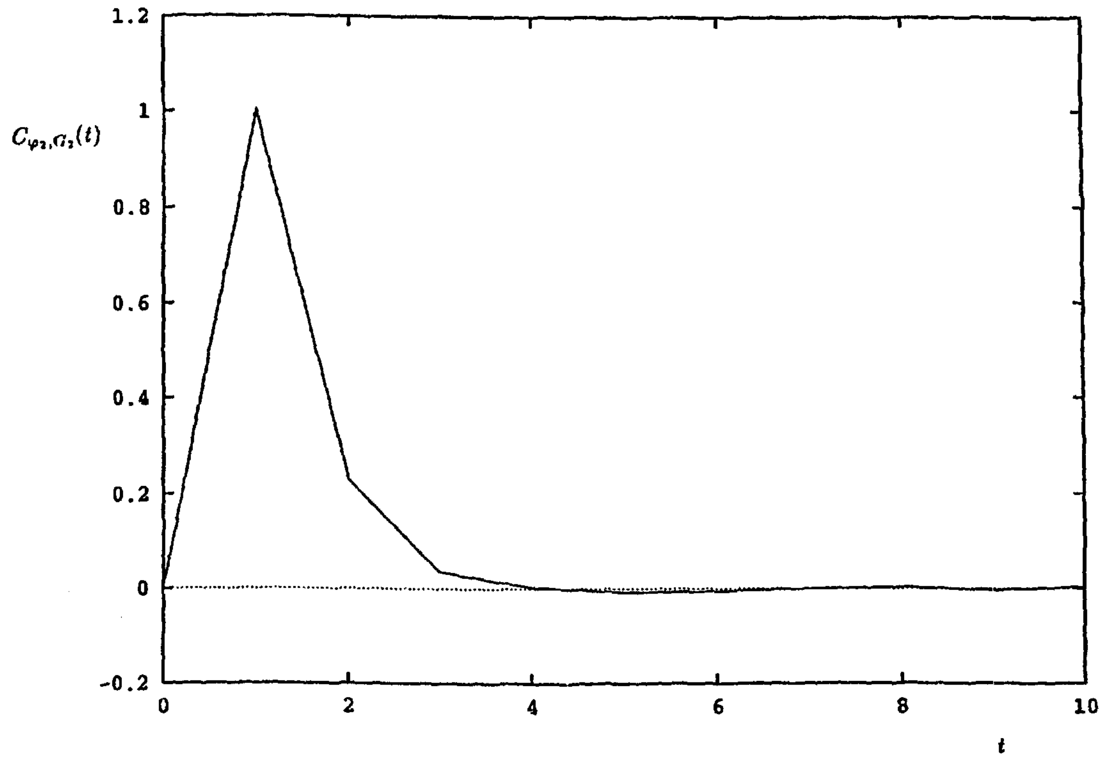

Figure 3. Numerical simulation of the correlation function $C_{\varphi_2,G_2}(t)$. The simulation was performed by direct calculation of the correlation function from trajectory iterates of the map.

This operator is also upper-triangular when acting on polynomials and has the same diagonal elements, including repeated eigenvalues as the map with Jordan block structure we have been discussing. But U_{T3} is completely diagonalizable, as recently obtained [10] using symmetry considerations [11]. The even-order eigenpolynomials of U_{T3} are Bernoulli polynomials and the odd-order eigenpolynomials are Euler polynomials.

Since U_{T3} is completely diagonalizable, the resolvent has only simple poles, in spite of the multiplicity of the eigenvalues. This can be verified using an expansion like (3) to study the matrix elements of the resolvent of U_{T3}. Because the codiagonal matrix elements, $\langle \tilde{B}_m | U_{T3} | B_{m+1} \rangle$, are all vanishing, no double poles occur.

3. CORRELATIONS WITH MODIFIED EXPONENTIAL DECAY

The Jordan block structure of U found in the previous section means that the even-order principal vectors evolve in time as

$$U^t \varphi_{2k}(x) = e^{-\gamma_{2k} t} \varphi_{2k}(x) + t e^{-\gamma_{2k}(t-1)} \varphi_{2k-1}(x).$$

In order to see the modified exponential decay on an observable, we may consider the correlation of $\varphi_{2k}(x)$ with another polynomial function $G(x)$ as

$$\begin{aligned} C_{\varphi_{2k},G}(t) &\equiv \lim_{T\to\infty} \frac{1}{T} \sum_{\tau=0}^{T-1} \varphi_{2k}(x_\tau) G(x_{t+\tau}) \\ &= \int_0^1 dx_0 \varphi_{2k}(x_0) G(x_t) \\ &= \int_0^1 dx \varphi_{2k}(x) \left(U^\dagger\right)^t G(x) \end{aligned}$$

$$= \int_0^1 dx U^t \varphi_{2k}(x) G(x)$$
$$= \int_0^1 dx \left[e^{-\gamma_{2k} t} \varphi_{2k}(x) + t e^{-\gamma_{2k}(t-1)} \varphi_{2k-1}(x) \right] G(x).$$

Then, by considering a certain function $G_{2k}(x)$, which is orthogonal to $\varphi_{2k}(x)$ but not to $\varphi_{2k-1}(x)$, the correlation would show just the $te^{-\gamma_{2k}(t-1)}$ behavior.

From the explicit forms of $\varphi_2(x)$ and $\varphi_1(x)$ given in the previous section, it is straightforward to obtain an appropriate $G_2(x)$ to be $-60(x^2 - (6/5)x)$, where the factor of -60 is put so that the value of the correlation at $t = 1$ is 1. A numerical simulation by a direct evaluation of the correlation $C_{\varphi_2, G_2}(t)$, from iterates of the map has been performed and the result is shown in Figure 3. Initial growth of the correlation is observed. It agrees with the predicted behavior of $t(1/9)^{t-1}$. The same correlation under evolution of the three-branch incomplete tent map would remain at its initial value of zero.

4. CONCLUDING REMARKS

The simplest example of a one-dimensional Markov map with a generalized spectral decomposition containing Jordan blocks has been given. Until now, systems for which the generalized spectral decomposition have been constructed have fallen into two classes. The first class contains two-dimensional maps with unitary Frobenius-Perron operators, which are diagonalizable in Hilbert space, but have a generalized spectral decomposition containing Jordan blocks. The second class consists of one-dimensional maps for which no spectral decomposition exists in Hilbert space [2–5] but which have diagonal generalized spectral decompositions. The one-dimensional map discussed here also does not have a spectral decomposition in Hilbert space, but its generalized spectral decomposition contains Jordan blocks; thus it falls into a new class.

REFERENCES

1. A. Lasota and M. Mackey, *Probabilistic Properties of Deterministic Systems*, Cambridge University Press, Cambridge, (1985).
2. I. Antoniou and S. Tasaki, Spectral decomposition of the Renyi map, *J. Phys. A* **26**, 73, (1993).
3. H.H. Hasegawa and W.C. Saphir, Unitarity and irreversibility in chaotic systems, *Phys. Rev. A* **46**, 7401, (1992).
4. P. Gaspard, r-adic one-dimensional maps and the Euler summation formula, *J. Phys. A* **25**, L483, (1992).
5. H.H. Hasegawa and D.J. Driebe, Intrinsic irreversibility and the validity of the kinetic description of chaotic systems, *Phys. Rev. E* **50**, 1781, (1994).
6. A. Baranovsky and D. Daems, Design of 1-D chaotic maps with prescribed statistical properties, *Intl. Journal of Bifurcation and Chaos* (to appear).
7. I. Antoniou and S. Tasaki, Generalized spectral decomposition of the β-adic baker's transformation and intrinsic irreversibility, *Physica A* **190**, 303, (1993).
8. D. Mayer and G. Roepstorff, On the relaxation time of Gauss' continued-fraction map. II, *J. Stat. Phys.* **50**, 331, (1988).
9. D.J. Driebe, *Nonequilibrium Statistical Mechanics of Fully Chaotic Maps*, Kluwer Academic, Dordrecht (to appear).
10. G.E. Ordóñez and D.J. Driebe, Spectral decomposition of tent maps using symmetry considerations, *J. Stat. Phys.* **84**, 269, (1996).
11. D.J. Driebe and G.E. Ordóñez, Using symmetries of the Frobenius-Perron operator to determine spectral decompositions, *Physics Letters A* **211**, 204, (1996).